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## A NEW CANONICAL FORM OF THE ELLIPTIC INTEGRAL

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The elliptic norm curve  $E_n$  in space  $S_{n-1}$  admits a group  $G_{2n^2}$  of collineations, and in fact there is a single infinity of such curves which admit the same group. A particular  $E_n$  of the family is distinguished by a value of the parameter  $\tau$ , itself an elliptic modular function defined by the modular group congruent to identity (mod  $n$ ).

In the group  $G_{2n^2}$  there are certain involutory collineations with two fixed spaces. If  $E_n$  is projected from one fixed space upon the other, a family of rational curves  $R_m$  mapping the family of  $E_n$ 's, is obtained. The quadratic irrationality separating involutory points on  $E_n$  involves the modulus  $\tau$  and the parameter  $t$  of the  $R_m$ . When the genus of the modular group is zero and  $n = 3, 4, 5$ , the irrationality can be used to define the elliptic parameter

$$u_1 = \int \frac{(t dt)}{(t \tau) \alpha_\tau^{r-3} \alpha_t^3},$$

where  $\alpha_t^r$  is the tetrahedral, octahedral, or icosahedral form. This is in contrast to Klein's form<sup>1</sup> as developed by Bianchi,<sup>2</sup> for there the normal elliptic integral is a rational curvilinear integral along an elliptic curve.

A comparison of the two integrals is more illuminating if it is carried out for a special case. Let  $E_n$  be  $E_5$  in  $S_4$ . In Bianchi's notation the five quadrics having  $E_5$  as their common intersection are

$$\varphi_i : a x_i^2 + a^2 x_{i+2} x_{i+3} - x_{i+1} x_{i+4} = 0, (x_{i+5} \equiv x_i), (i = 0, \dots, 4),$$

where  $a$  is the modulus. If a transformation of coordinates is made in order to bring into evidence the fixed spaces of the involutory collineation used in the projection, then the icosahedral form which appears in the irrationality is

$$\alpha_t^{12} = t_1 t_2 (t_1^{10} + 11 t_1^5 t_2^5 - t_2^{10}).$$

The integral  $u_1$  involving  $\tau = a$  explicitly in a rather simple form is uniquely defined. Moreover it is invariant under all cogredient trans-

formations of  $t$  and  $\tau$ , which leave the form  $\alpha_x^{19}$  unaltered, i.e., the sixty transformations of the icosahedral group applied simultaneously to  $t$  and  $\tau$ , the parameter of the doubly-covered conic  $R_2$  and the modulus of the elliptic quintic curve  $E_5$ , leave  $u_1$  unaltered.

Consider now Bianchi's integral. It is defined as

$$U = C \int \frac{(u \, dv - v \, du)}{(\varphi_0 \varphi_1 \varphi_2 u \, v)},$$

where  $C$  is a constant,  $u$  and  $v$  any two expressions linear in  $x$ , and the denominator is the functional determinant of  $\varphi_0, \varphi_1, \varphi_2, u$  and  $v$ . For a particular choice of  $u$  and  $v$  the integral assumes the simple form

$$U = C \int \frac{(x_0 \, dx_1 - x_1 \, dx_0)}{5 \, a^3 \, x_2 \, x_4 - (2 \, a^5 + 1) \, x_0 \, x_1},$$

where the  $x$ 's are subject to the relations  $\varphi_i$ . Different expressions for  $U$  can be obtained by making different choices for  $u$  and  $v$ . Hence there is no unique form for  $U$  as there is for  $u_1$ . The integral  $U$  assumes various conjugate forms under the Group  $G_{50}$  of collineations on the  $x$ 's, and also under the transformations of  $a$ .

So the integral  $u_1$  seems to have an advantage over  $U$  in its simplicity of form, its uniqueness, and its invariancy under transformations.

By a study of the integral  $u_1$  itself some interesting results are derived. The modular equation connecting  $\tau$  and  $J$ , the absolute invariant of  $u_1$ , can be deduced as the result of the binary syzygy of lowest weight connecting the concomitants of  $\alpha_x^7$ . The requirement that the Riemann surface attached to the modular equation be regular leads to the modular equations associated with the regular bodies. It is then possible to eliminate the more tedious individual proofs used by Bianchi in the discussion of the moduli of  $E_3$  and  $E_6$  to show that these moduli are the tetrahedral and icosahedral irrationalities respectively. In fact the algebraic discussion carried out once for  $\alpha_x^7$  is complete for factor groups of genus zero, which have been discussed by Klein,<sup>3</sup> i.e., those isomorphic with the groups associated with the regular bodies, namely, the one dihedral group  $G_6$  and the tetrahedral, octahedral, and icosahedral groups.

<sup>1</sup> Klein-Fricke, *Vorlesungen über die Theorie der Elliptischen Modulfunctionen*, Bd. 2, Abschnitt 5.

<sup>2</sup> Bianchi, Über die Normalformen dritter und fünfter Stufe des elliptischen Integrals. erster Gattung, *Math. Ann.*, Leipzig, 17, 234-262, (1880).

<sup>3</sup> Klein-Fricke, loc. cit., vol. 1, pp. 339 ff.